Linear Transformations (or, Matrices again)

In calculus, you deal with functions $f: \mathbb{R} \to \mathbb{R}$ (1-dimensional vectors)

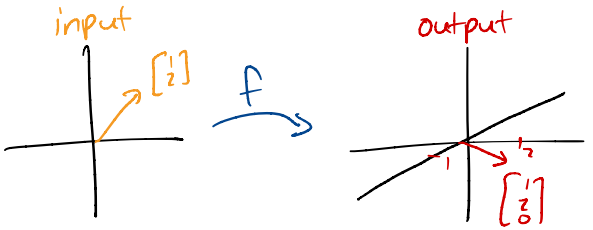
E.g. $f(x) = x^2$, $g(x) = \sin x$

We will consider functions $f: \mathbb{R}^n \to \mathbb{R}^m$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \\ 2x-y \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$



A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

- $T(v+w) = T(v) + T(w) \quad \forall\, v, w \in \mathbb{R}^n$
- $T(cv) = cT(v) \quad \forall\, v \in \mathbb{R}^n,\ c \in \mathbb{R}$

Ex: Functions from calculus:

a) $f(x) = x^2$ is not linear since $f(1+2) = (1+2)^2 = 9$
while $f(1) + f(2) = 1^2 + 2^2 = 5$

b) $g(x) = 3x$ is linear since:

- $g(x+y) = 3(x+y) = 3x + 3y = g(x) + g(y)$
- $g(cx) = 3(cx) = 3cx = c(3x) = cg(x)$

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x \\ -2y \end{bmatrix}$ is a linear transformation:

Verify in Recitation

Ex: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x^2 \\ y \end{bmatrix}$ is not a linear transformation since

$2 \cdot T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

but $T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 12 \\ 2 \end{bmatrix}$

So $2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Heuristically, $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if the entries of the output vector look linear (ie. they are sums of multiples of variables)

Key Point: The output vector can be rewritten as a linear combination whose weights are the inputs, aka Matrix-Vector Multiplication

Ex: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x \\ -2y \end{bmatrix} = x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -2 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

So $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$

evaluation = Matrix-Vector Multiplication

In general, a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be encoded by an $m \times n$ matrix A such that

$$T(v) = Av \quad \forall \; v \in \mathbb{R}^n$$

Ex: $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(v) = v$ is called the identity map. It has associated matrix I_n since $T(v) = v = I_n v$.

Given two linear transformations S and T, we can often create new functions:

$S+T$, cT, ST, $\frac{S}{T}$, $S \circ T$, S^{-1} (if they exist)

ST, $\frac{S}{T}$: not linear

e.g. $S(x) = 2x$, $T(x) = 3x$ are linear but $(S \cdot T)(x) = 6x^2$ is not linear

Let A be the matrix associated to S and B be the matrix associated to T

Then:

- $S+T$ is encoded by $A+B$
- cS is encoded by cA
- $S \circ T$ is encoded by AB
- S^{-1} is encoded by A^{-1}

(if they are defined)

Will think about this in recitation.

This is why matrix operations are defined as they are.

Systems of Equations

Question: Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$. Does there exist $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$?

or equivalently, solve $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (matrix equation)

$A \quad x \quad = \quad b$

or equivalently, solve $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (L.C. eqn)

or equivalently, solve $\begin{bmatrix} x-y \\ 2x+3y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (vector eqn)

or equivalently, solve $\begin{aligned} x - y &= 1 \\ 2x + 3y &= 3 \end{aligned}$ (system of eqns)

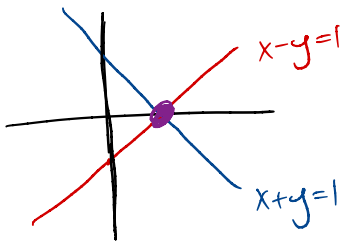
Since this system has solution $x = 6/5$, $y = 1/5$ (Check)

- $T\left(\begin{bmatrix} 6/5 \\ 1/5 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$,
- $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a L.C. of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ with weights $6/5$ and $1/5$
- $Ax = b$ has sol'n $\begin{bmatrix} 6/5 \\ 1/5 \end{bmatrix}$.

Geometrically solving a systems corresponds to looking for points of intersection.

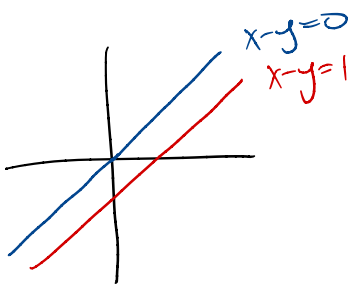
In $\mathbb{R}^2$: lines

- $x-y=1$
  $x+y=1$



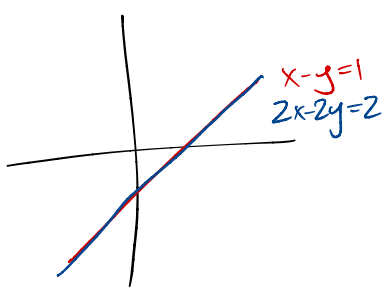
one solution

- $x-y=1$
  $x-y=0$



no solutions

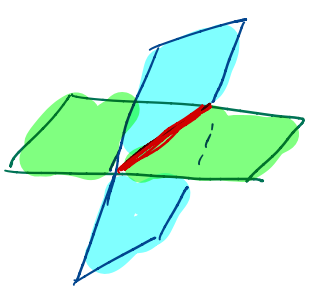
- $x-y=1$
  $2x-2y=2$



infinitely many solutions

In $\mathbb{R}^3$: Planes

2 eqns:



Will play with 3D pictures in recitation.

A system of linear equations can have 0, 1, or infinitely many solutions (and no other possibilities)